Kiril Datchev
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Suggested topics for the final project

For the final project, write a short paper on a topic of your choice related to PDEs, explaining one or more important results and/or examples, at a level appropriate to our class, and prepare a 30 minute presentation based on this to give in the last weeks of the semester. Be sure to include lots of pictures, some technical details (although you may also omit some), and a few references.

You may choose one of the suggested topics below or come up with your own. Proposing your own topic is especially encouraged if you have some background or interest from outside the course which you would like to share with us. Please let me know if you are having any difficulty and I will try to help. I've included some possible starting points for reading about the topics below, but you are also encouraged to seek out your own references to supplement or replace the ones given.

(1) First order partial differential equations. In class we developed the method of characteristics for linear equations and some nonlinear ones. The method applies in a generality analogous to the one for the local theory of ordinary differential equations. A good source is Section 3.2 of Evans' book [4], with the main result being the solution obtained in Theorem 2 there: this is the PDE version of the fundamental ODE theorem.

(2) The solution of the vibrating membrane problem for a circular drum can be written in terms of Bessel functions, as discussed in Section 5.3 of Borthwick's book [1]. The same applies to the heat equation in such a domain, and also to balls in 3 or more dimensions. The important properties of these functions are derived to some extent in Chapter 10 of Strauss' book [9] and more fully in Chapters 15 and 16 of Vasy's book [11]. The major mathematical techniques used are series solutions of ODEs and the principle of stationary phase/steepest descent for integration. Dimension 3 is significantly simpler than dimension 2.

(3) More general waves than the ones we studied have a finite speed of propagation, and this can be proved by energy methods. See Section 9.1 of Strauss' book [9] and Chapter 7 of Vasy's book [11] especially Problem 7.3.

(4) We saw how energy estimates can be broadly used to prove uniqueness of solutions to equations. Using an appropriate concept of duality, they can also be used to prove existence of solutions and construct those solutions. This is the infinite dimensional version of the linear algebra fact that injectivity of a linear map implies surjectivity of the dual map (transpose or adjoint). Some of this theory is developed and implemented for elliptic equations in Section 7.4 and Chapter 17 of Vasy's book [11].

(5) Spherical harmonics are eigenfunctions of the Laplacian on the sphere, and are important in the study of the Laplacian in polar coordinates, with applications to the electronic structure of atoms. The basics of spherical harmonics are developed in [3], and details can be found in Section 7.4 of Taylor's book [10].

(6) One of the reasons that pointwise convergence is hard to work with is that it does not correspond to any norm. This follows from some general results in topology (see Example 2 of Section 21 and Theorem 46.1 of Munkres' book [5]) but it should be possible to explain it more directly.

(7) Calderón's Inverse Problem asks: when can the electrical conductivity of an object be recovered from boundary measurements of voltage and current? A good starting point is Calderón's original paper [2]; see also Section 3 of Salo's notes [6].

(8) The Fourier transform can be used to invert X-ray and Radon transforms. These are important in imaging problems, in which we wish to recover a function from certain integrals of the function. See Section 8.5 of Vasy's book [11]. Another reference, in which this is done in terms of fractional powers of the Laplace operator, is Section 5.2 and Problem 8 of Chapter 6 of Stein and Shakarchi's book [8].

(9) Some nonlinear evolution equations can be solved by a contraction mapping argument, analogous to how one proves the fundamental theorem of ODEs. You can see a version of this for waves in Theorem 4 of Shao's notes [7]. To make things simpler, you can consider Sobolev spaces of integer order, so that the estimates are easier and can be done using the defition (10.17) of [1].

References

[1] David Borthwick, *Introduction to Partial Differential Equations.* Springer, 2016.

[2] Alberto P. Calderón. *On an inverse boundary value problem.* Seminar on Numerical Analysis and its Applications to Continuum Physics (Rio de Janeiro, 1980), pp. 65–73, Soc. Brasil. Mat., Rio de Janeiro, 1980. Reprinted in Comput. Appl. Math. 25 (2006), no. 2-3, 133–138. `https://www.scielo.br/j/cam/a/fr8pXpGLSmDt8JyZyxvfwbv/?format=pdf&lang=en`.

[3] Kiril Datchev, *The Laplacian in polar coordinates and spherical harmonics*, `https://www.math.purdue.edu/~kdatchev/495%7c595/polar.pdf`.

[4] Lawrence C. Evans, *Partial Differential Equations.* AMS Graduate Studies in Mathematics, Volume 19, Second Edition, 2010.

[5] James R. Munkres, *Topology.* Prentice Hall, Second Edition, 2000

[6] Mikko Salo, *Calderón problem.* `http://users.jyu.fi/~salomi/lecturenotes/calderon_lectures.pdf`.

[7] Arick Shao, *Nonlinear wave equations – classical methods* `https://lpde.maths.qmul.ac.uk/a.shao/research/notes/wave_cl.pdf`.

[8] Elias M. Stein and Rami Shakarchi. *Fourier Analysis*, Princeton University Press, 2003.

[9] Walter A. Strauss, *Partial Differential Equations.* Wiley, Second Edition, 2009.

[10] Michael E. Taylor, *Introduction to Analysis in Several Variables*, AMS Sally Series of Pure and Applied Undergraduate Texts 46, 2020. Preprint available online at `https://mtaylor.web.unc.edu/wp-content/uploads/sites/16915/2018/04/analmv.pdf`.

[11] András Vasy, *Partial Differential Equations.* AMS Graduate Studies in Mathematics, Volume 169, 2015.